The bicovariant differential calculus on the $\kappa-$Poincarè and $\kappa-$Weyl groups

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Abstract

The bicovariant differential calculus on four-dimensional $\kappa-$Poincare group and corresponding Lie-algebra like structure for any metric tensor are described. The bicovariant differential calculus on four-dimensional $\kappa-$Weyl group and corresponding Lie-algebra like structure for any matric tensor in the reference frame in which $g_{00} = 0$ are considered.

1 Introduction

Recently, considerable interest has been paid to the deformations of groups and algebras of space-time symmetries [7]. In particular, an interesting deformation of the Poincaré algebra [8] as well as group [9] has been introduced which depend on dimensionful deformation parameter κ; the relevant objects are called κ-Poincaré algebra and κ-Poincaré group, respectively. Their structure was studied in some detail and many of their properties are now well understood. The κ-Poincaré algebra and group for the space-time of any dimension has been defined [10] , the realizations of the algebra in terms of differential operators acting on commutative Minkowski as well as momentum spaces were given [11]; the unitary representations of the deformed group were found [12]; the deformed universal covering $ISL(2, \mathcal{C})$ was constructed [13]; the bicrossproduct [14] structure, both of the algebra and group was revealed [15]. The proof of formal duality between κ-Poincaré group and κ-Poincaré algebra was also given, both in two [16] as well as in four dimensions [17]. One of the important problems is the construction of the bicovariant differential calculus on κ-Poincaré group. Using an elegant approach due to Woronowicz [5], the differential calculi on four-dimensional Poincaré group for diagonal metric tensor [2] and three-dimensional [3], as well as on the Minkowski space [2] and [4] were constructed.

In the paper [1] the $\kappa-$deformation of the Poincarè algebra and group for arbitrary metric tensor has been described and under the assumption that $g_{00} = 0$ the $\kappa-$deformation of the Weyl group as well as algebra has been constructed.

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In this paper in section 2 we briefly sketch the construction of differential calculus on the $\kappa-$Poincarè group for any metric tensor $g_{\mu\nu}, \quad \mu,\,\nu = 0, ..., 3$. We obtain the corresponding Lie algebra structure and prove its equivalence to the $\kappa-$Poincarè algebra. In section 4 we present the construction of differential calculus on the $\kappa-$Weyl group for any metric tensor $g_{\mu\nu}, \quad \mu,\,\nu = 0, ..., 3$ with $g_{00} = 0$. We find the corresponding Lie algebra structure and prove its equivalence to the $\kappa-$Weyl algebra.

2 The $\kappa-$Poincarè group and algebra

We assume that the metric tensor $g_{\mu\nu}, (\mu\nu = 0, 1, ..., 3)$ is represented by an arbitrary nondegenerate symetric 4×4 matrix (not necessery diagonal) with $det(g_{\mu\nu}) = 1$ (in more general case in some equations we use the parameter $det(g)$).

The Poincarè group $\mathcal{P}$ consists of the pairs (x, Λ), where x is a $4-$vector, Λ is the matrix of the Lorentz group in $4-$dimensions, with the composition law:

$$(x^\mu, \Lambda^\mu{}_\nu) * (x'^\nu, \Lambda'^\nu{}_\alpha) = (\Lambda^\mu{}_\nu x'^\nu + x^\mu, \Lambda^\mu{}_\nu \Lambda'^\nu{}_\alpha).$$

The $\kappa-$Poincarè group is a Hopf $*-$algebra defined as follows [1]. Consider the universal $*-$algebra with unity, generated by selfadjoint elements $\Lambda^\mu{}_\nu,\ x^\mu$ subject to the following relations:

$$\begin{aligned}
[\Lambda^\alpha{}_\beta, x^\varrho] &= -\frac{i}{\kappa}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\varrho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\varrho}),\\
[x^\varrho, x^\sigma] &= \frac{i}{\kappa}(\delta^\varrho{}_0 x^\sigma - \delta^\sigma{}_0 x^\varrho),\\
[\Lambda^\alpha{}_\beta, \Lambda^\mu{}_\nu] &= 0.
\end{aligned}$$

The comultiplication, antipode and counit are defined as follows:

$$\begin{aligned}
\Delta\Lambda^\mu{}_\nu &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu,\\
\Delta x^\mu &= \Lambda^\mu{}_\nu \otimes x^\nu + x^\mu \otimes I,\\
S(\Lambda^\mu{}_\nu) &= \Lambda_\nu{}^\mu,\\
S(x^\mu) &= -\Lambda_\nu{}^\mu x^\nu,\\
\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu,\\
\varepsilon(x^\mu) &= 0.
\end{aligned}$$

In our construction of the bicovariant $*-$calculi we use the Woronowicz theory, [5]. First we construct the right ad-invariant ideal $\mathcal{R}$ in $ker\ \varepsilon,\ (ad(\mathcal{R}) = \mathcal{R} \otimes \mathcal{P}_\kappa)$. The adjoint action of the group is defined as follows:

$$ad(a) = \sum_k b_k \otimes S(a_k)c_k,$$

where

$$(\Delta \otimes I)\Delta(a) = (I \otimes \Delta)\Delta(a) = \sum_k a_k \otimes b_k \otimes c_k.$$

In order to obtain the ideal $\mathcal{R}$ we put:

$$\begin{aligned}
\Delta^\mu{}_\nu &= \Lambda^\mu{}_\nu - \delta^\mu{}_\nu, \\
\Delta^\mu{}_\nu{}^\alpha &= x^\alpha \Delta^\mu{}_\nu + \frac{i}{\kappa}(g_{\nu 0}\Delta^{\alpha\mu} - \delta^\mu{}_0 \Delta^\alpha{}_\nu), \\
x^{\alpha\beta} &= x^\alpha x^\beta + \frac{i}{\kappa}(g^{\alpha\beta}x_0 - \delta^\alpha{}_0 x^\beta), \\
\tilde{\Delta}^\mu{}_\nu{}^\alpha &= \Delta^\mu{}_\nu{}^\alpha - \boxed{\tfrac{1}{6}\varepsilon^\mu{}_\nu{}^{\alpha\gamma}\varepsilon_{\gamma\varrho\sigma\delta}\Delta^{\varrho\sigma\delta}}, \\
\tilde{x}^{\alpha\beta} &= x^{\alpha\beta} - \boxed{\tfrac{1}{4}g^{\alpha\beta}x^\mu{}_\mu}, \\
T &= lin\{\ \tilde{\Delta}^\mu{}_\nu{}^\alpha,\ \Delta^\alpha{}_\beta \Delta^\mu{}_\nu,\ \tilde{x}^{\alpha\beta}\ \}.
\end{aligned}$$

One can prove that the linear set T is ad-invariant. Let $\mathcal{R}$ be a right ideal of $\mathcal{P}_\kappa$ generated by elements of T. Then the following theorem holds:

Theorem 1. $\mathcal{R}$ has the following properties:

(i) $\mathcal{R}$ is ad-invariant,

(ii) for any $a \in \mathcal{R},\ S(a)^* \in \mathcal{R}$,

(iii) $ker\ \varepsilon/\mathcal{R}$ is spanned by the following elements:

$$x^\mu; \quad \Delta^\mu{}_\nu,\ \mu < \nu; \quad \varphi = x^\mu{}_\mu;$$

$$\varphi_\mu = \varepsilon_{\mu\varrho\sigma\delta}\Delta^{\varrho\sigma\delta}.$$

Note only that:

(a) $\Delta^\mu{}_\nu{}^\alpha,\ x^{\mu\nu}$ are "improved" generators of $(ker\ \varepsilon)^2$ which form the (not completely reducible) multiplet under adjoint action of $\mathcal{P}_\kappa$,

(b) if $g_{00} \neq 0$ then the ideal generated by $\Delta^\alpha{}_\beta\Delta^\mu{}_\nu,\ \Delta^\mu{}_\nu{}^\alpha,\ x^{\mu\nu}$ equals $ker\ \varepsilon$. In order to obtain reasonable (in the sesne that it contains all differentials $dx^\mu,\ d\Lambda^\mu{}_\nu$) calculus we have subtracted the trace of $x^{\mu\nu}$ and completely antisymmetric part of $\Delta^\mu{}_\nu{}^\alpha$.

(c) it is easy to conclude from (iii) that our calculus is fifteen-dimensional.

Now, following Woronowicz's paper [5] we find the basis of the space of the left-invariant 1-forms:

$$\begin{aligned}
\omega^\mu &= \pi r^{-1}(I \otimes x^\mu) = \Lambda_\nu{}^\mu dx^\nu, \\
\omega^\mu{}_\nu &= \pi r^{-1}(I \otimes \Delta^\mu{}_\nu) = \Lambda_\alpha{}^\mu d\Lambda^\alpha{}_\nu, \\
\Omega &= \pi r^{-1}(I \otimes \varphi) = d\varphi - 2x_\alpha dx^\alpha, \\
\Omega_\mu &= \pi r^{-1}(I \otimes \varphi_\mu) = \varepsilon_{\mu\nu\alpha\beta}\Lambda_\delta{}^\nu \omega^\beta \Lambda^{\delta\alpha} - \frac{2i}{\kappa}\varepsilon_{0\mu\nu\beta}\omega^{\nu\beta}.
\end{aligned}$$

The commutations rules between the invariant forms and generators of $\mathcal{P}_\kappa$ read:

$$\begin{aligned}
[\Lambda^{\mu}{}_{\nu}, \omega^{\alpha}] &= -\frac{i}{\kappa}(g_{\nu 0}\Lambda^{\mu}{}_{\tau}\omega^{\tau\alpha} + \Lambda^{\mu}{}_{0}\omega^{\alpha}{}_{\nu}) - \frac{1}{6}\varepsilon^{\tau}{}_{\nu}{}^{\alpha\gamma}\Lambda^{\mu}{}_{\tau}\Omega_{\gamma}, \\
[x^{\mu}, \omega^{\alpha}] &= -\frac{1}{4}\Lambda^{\mu\alpha}\Omega + \frac{i}{\kappa}(\Lambda^{\mu\alpha}\omega_0 - \delta^{\alpha}{}_{0}\Lambda^{\mu}{}_{\nu}\omega^{\nu}), \\
[\Lambda^{\alpha}{}_{\beta}, \omega^{\mu}{}_{\nu}] &= 0, \\
[x^{\alpha}, \omega^{\mu}{}_{\nu}] &= -\frac{i}{\kappa}(\delta^{\mu}{}_{0}\Lambda^{\alpha}{}_{\beta}\omega^{\beta}{}_{\nu} + g_{\nu 0}\Lambda^{\alpha}{}_{\beta}\omega^{\mu\beta} - \Lambda^{\alpha}{}_{\nu}\omega^{\mu}{}_{0} - \Lambda^{\alpha\mu}\omega_{0\nu}) - \frac{1}{6}\varepsilon^{\mu}{}_{\nu}{}^{\beta\gamma}\Lambda^{\alpha}{}_{\beta}\Omega_{\gamma}, \\
[\Lambda^{\mu}{}_{\nu}, \Omega] &= \frac{4}{\kappa^2}g_{00}\Lambda^{\mu\tau}\omega_{\tau\nu}, \\
[x^{\alpha}, \Omega] &= \frac{4}{\kappa^2}g_{00}\Lambda^{\alpha}{}_{\beta}\omega^{\beta}, \\
[\Lambda^{\alpha}{}_{\beta}, \Omega_{\mu}] &= 0, \\
[x^{\alpha}, \Omega_{\mu}] &= \frac{3}{\kappa^2}g_{00}\varepsilon_{\mu\nu\tau\beta}\Lambda^{\alpha\nu}\omega^{\tau\beta} + \frac{i}{\kappa}(\Lambda^{\alpha}{}_{\mu}\Omega_0 - g_{0\mu}\Lambda^{\alpha\beta}\Omega_{\beta}).
\end{aligned}$$

The corresponding basis of the space of the right-invariant 1-forms is:

$$\begin{aligned}
\eta^{\mu} &= -\omega^{\alpha}{}_{\beta}\Lambda_{\nu}{}^{\beta}x^{\nu}\Lambda^{\mu}{}_{\alpha} + \omega^{\alpha}\Lambda^{\mu}{}_{\alpha}, \\
\eta^{\mu}{}_{\nu} &= \omega^{\alpha}{}_{\beta}\Lambda^{\mu}{}_{\alpha}\Lambda_{\nu}{}^{\beta}, \\
\theta &= \Omega, \\
\theta_{\mu} &= \Omega_{\nu}\Lambda_{\mu}{}^{\nu}.
\end{aligned}$$

This concludes the description of bimodule Γ of 1-forms on $\mathcal{P}_{\kappa}$. The external algebra can be now constructed as follows [5]. On $\Gamma^{\otimes 2}$ we define a bimodule homomorpfism σ such that

$$\sigma(\omega \otimes_{\mathcal{P}_{\kappa}} \eta) = \eta \otimes_{\mathcal{P}_{\kappa}} \omega,$$

for any left-invariant $\omega \in \Gamma$ and any right-invariant $\eta \in \Gamma$. Then by the definition

$$\Gamma^{\wedge 2} = \frac{\Gamma^{\otimes 2}}{ker(I - \sigma)}.$$

Finally, after a long analysis we obtain the following set of relations:

$$\begin{aligned}
&\Omega \wedge \Omega = 0, \\
&\omega^{\mu}{}_{\nu} \wedge \omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} \wedge \omega^{\mu}{}_{\nu} = 0, \\
&\Omega_{\alpha} \wedge \Omega_{\beta} + \Omega_{\beta} \wedge \Omega_{\alpha} = 0, \\
&\Omega_{\alpha} \wedge \omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \wedge \Omega_{\alpha} = 0, \\
&\Omega \wedge \omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \wedge \Omega - \frac{4}{\kappa^2}g_{00}\omega_{\beta\nu} \wedge \omega^{\beta\mu} = 0, \\
&\Omega_{\mu} \wedge \Omega + \Omega \wedge \Omega_{\mu} - \frac{4}{\kappa^2}g_{00}\Omega_{\tau} \wedge \omega^{\tau}{}_{\mu} - \frac{4}{\kappa^2}g_{00}\mathcal{X}_{\mu} = 0, \\
&\omega^{\mu} \wedge \Omega + \Omega \wedge \omega^{\mu} - \frac{4}{\kappa^2}g_{00}\omega^{\mu}{}_{\beta} \wedge \omega^{\beta} = 0,
\end{aligned}$$

$$
\begin{aligned}
&\omega^\alpha \wedge \omega^\mu + \omega^\mu \wedge \omega^\alpha + \frac{i}{\kappa}(\delta^\alpha_{\ 0}\omega^\mu_{\ \beta} \wedge \omega^\beta + \delta^\mu_{\ 0}\omega^\alpha_{\ \beta} \wedge \omega^\beta) = 0,\\
&\omega^\alpha \wedge \omega^{\mu\nu} + \omega^{\mu\nu} \wedge \omega^\alpha + \frac{i}{\kappa}(\delta^\mu_{\ 0}\omega^{\tau\nu} \wedge \omega_\tau^{\ \alpha} + \delta^\nu_{\ 0}\omega^\mu_{\ \tau} \wedge \omega^{\tau\alpha} + \omega_0^{\ \nu} \wedge \omega^{\alpha\mu} + \omega^\mu_{\ 0} \wedge \omega^{\alpha\nu})\\
&-\frac{1}{6}\varepsilon^{\alpha\mu\nu\sigma}\mathcal{X}_\sigma = 0,\\
&\omega^\mu \wedge \Omega^\alpha + \Omega^\alpha \wedge \omega^\mu + \frac{i}{\kappa}(\delta^\alpha_{\ 0}\Omega_\tau \wedge \omega^{\tau\mu} + \Omega_0 \wedge \omega^{\mu\alpha}\\
&+\frac{1}{12}\varepsilon^{\alpha\mu\tau\sigma}\Omega_\tau \wedge \Omega_\sigma - \frac{3}{2\kappa^2}g_{00}\varepsilon^{\alpha\sigma\tau\beta}\omega^\mu_{\ \sigma} \wedge \omega_{\tau\beta} - \frac{1}{4}g^{\alpha\mu}\mathcal{Y} = 0,
\end{aligned}
$$

where we introduce the following 2-forms:

$$
\begin{aligned}
\mathcal{Y} &= \omega_\alpha \wedge \Omega^\alpha + \Omega^\alpha \wedge \omega_\alpha + \frac{i}{\kappa}\Omega_\tau \wedge \omega^\tau_{\ 0},\\
\mathcal{X}_\sigma &= \varepsilon_{\sigma\alpha\mu\nu}(\omega^\alpha \wedge \omega^{\mu\nu} + \omega^{\mu\nu} \wedge \omega^\alpha + \frac{2i}{\kappa}(\delta^\mu_{\ 0}\omega^{\tau\nu} \wedge \omega_\tau^{\ \alpha} + \omega_0^{\ \nu} \wedge \omega^{\alpha\mu})).
\end{aligned}
$$

The basis of $\Gamma^{\wedge 2}$ consists of the following elements:

$$
\begin{aligned}
&\omega^{\alpha\beta} \wedge \omega^{\mu\nu}; \quad \text{for } \alpha < \beta,\ \mu < \nu,\ (\alpha\beta) \neq (\mu\nu),\ \alpha < \mu;\\
&\Omega^\mu \wedge \Omega^\nu,\ \omega^\mu \wedge \Omega^\nu,\ \omega^{\mu\nu} \wedge \Omega^\alpha,\ \omega^{\mu\nu} \wedge \Omega,\ \omega^\mu \wedge \omega^\nu,\ \omega^{\mu\nu} \wedge \omega^\alpha; \quad \text{for } \mu < \nu;\\
&\omega^\mu \wedge \Omega,\ \Omega^\mu \wedge \Omega,\ \mathcal{Y},\ \mathcal{X}^\mu.
\end{aligned}
$$

Thus, there are five more elements than it is generically expected. The Cartan-Maurer equations have the following form:

$$
\begin{aligned}
d\omega^\mu_{\ \nu} &= \omega_\tau^{\ \mu} \wedge \omega^\tau_{\ \nu},\\
d\omega^\mu &= \omega_\tau^{\ \mu} \wedge \omega^\tau,\\
d\Omega &= 0,\\
d\Omega_\mu &= -\omega_\mu^{\ \alpha} \wedge \Omega_\alpha - \mathcal{X}_\mu.
\end{aligned}
$$

To obtain the quantum Lie algebra we introduce the left-invariant fields, defined by the formula:

$$
da = \frac{1}{2}(\chi_{\mu\nu} * a)\omega^{\mu\nu} + (\chi_\mu * a)\omega^\mu + (\chi * a)\Omega + (\lambda_\mu * a)\Omega^\mu, \tag{2.1}
$$

where, for any linear functional φ on $\mathcal{P}_\kappa$,

$$
\varphi * a = (I \otimes \varphi)\Delta(a).
$$

The product of two functional $\varphi_1,\ \varphi_2$ is defined by the duality relation:

$$
\varphi_1 * \varphi_2(a) = (\varphi_1 \otimes \varphi_2)\Delta(a).
$$

Finally, we apply the external derivative to both sides of eq.(2.1). Using the fact that $d^2a = 0$ and equating to zero the coefficients of the basis elements of $\Gamma^{\wedge 2}$ we find the quantum Lie algebra:

$$\begin{aligned}
\lambda^\mu \chi_\mu &= 0, \\
[\chi_{\mu\nu}, \chi] &= 0, \\
\lambda_\mu(\frac{4}{\kappa^2} g_{00}\chi - 1) &= \frac{1}{12} \varepsilon_\mu{}^{\alpha\beta\nu} \chi_\alpha \chi_{\beta\nu}, \\
[\chi_{\mu\nu}, \chi_\alpha] &= (1 + \frac{i}{\kappa}\chi_0 - \frac{4}{\kappa^2} g_{00}\chi)(\chi_\mu g_{\alpha\nu} - \chi_\nu g_{\alpha\mu}) \\
[\lambda_\mu, \chi] &= 0, \\
[\chi_\alpha, \chi_\mu] &= 0 \\
[\chi, \chi_\mu] &= 0, \\
[\chi_{\mu\nu}, \lambda_\alpha] &= (\lambda_\mu g_{\alpha\nu} - \lambda_\nu g_{\alpha\mu})(1 - \frac{4}{\kappa^2} g_{00}\chi) \\
&\quad - \frac{i}{\kappa}\lambda_0(\chi_\nu g_{\mu\alpha} - \chi_\mu g_{\nu\alpha}) - \frac{i}{\kappa} g_{0\alpha}(\lambda_\nu \chi_\mu - \lambda_\mu \chi_\nu), \\
[\lambda_\alpha, \chi_\mu] &= 0, \\
[\lambda_\alpha, \lambda_\mu] &= -\frac{i}{6} \varepsilon_{\mu\alpha}{}^{\sigma\delta} \lambda_\sigma \chi_\delta, \\
[\chi_{\alpha\beta}, \chi_{\mu\nu}] &= (1 - \frac{4}{\kappa^2} g_{00}\chi)(\chi_{\beta\mu} g_{\nu\alpha} + \chi_{\alpha\nu} g_{\mu\beta} - \chi_{\beta\nu} g_{\mu\alpha} - \chi_{\alpha\mu} g_{\nu\beta}) \\
&\quad + \frac{i}{\kappa}(\chi_\alpha(\chi_{\mu 0} g_{\nu\beta} - \chi_{\nu 0} g_{\mu\beta}) + \chi_\mu(\chi_{\beta 0} g_{\alpha\nu} - \chi_{\alpha 0} g_{\nu\beta}) \\
&\quad + \chi_\nu(\chi_{\alpha 0} g_{\mu\beta} - \chi_{\beta 0} g_{\alpha\mu}) + \chi_\beta(\chi_{\nu 0} g_{\alpha\mu} - \chi_{\mu 0} g_{\alpha\nu})) \\
&\quad + \frac{i}{\kappa}(\chi_\beta(\chi_{\mu\alpha} g_{0\nu} - \chi_{\nu\alpha} g_{0\mu}) + \chi_\nu(\chi_{\beta\mu} g_{0\alpha} - \chi_{\alpha\mu} g_{0\beta}) \\
&\quad + \chi_\mu(\chi_{\alpha\nu} g_{0\beta} - \chi_{\beta\nu} g_{0\alpha}) + \chi_\alpha(\chi_{\nu\beta} g_{0\mu} - \chi_{\mu\beta} g_{0\nu})) \\
&\quad + \frac{3}{\kappa^2} g_{00} \lambda_\sigma(\chi_\beta \varepsilon^\sigma{}_{\alpha\mu\nu} - \chi_\alpha \varepsilon^\sigma{}_{\beta\mu\nu} + \chi_\mu \varepsilon^\sigma{}_{\nu\alpha\beta} - \chi_\nu \varepsilon^\sigma{}_{\mu\alpha\beta}).
\end{aligned}$$

Having our quantum Lie algebra constructed, we can now pose the question what the relation is between our functionals and the elements of the $\kappa-$Poincare algebra $\tilde{\mathcal{P}}_\kappa$.

The $\kappa-$Poincare algebra is defined as follows [1], [17]:
The commutation rules:

$$\begin{aligned}
[M^{ij}, P_0] &= 0, \\
[M^{ij}, P_k] &= i\kappa(\delta^j{}_k g^{0i} - \delta^i{}_k g^{0j})(1 - e^{-\frac{P_0}{\kappa}}) + i(\delta^j{}_k g^{is} - \delta^i{}_k g^{js})P_s, \\
[M^{i0}, P_0] &= i\kappa g^{i0}(1 - e^{-\frac{p_0}{\kappa}}) + i g^{ik} P_k, \\
[M^{i0}, P_k] &= -i\frac{\kappa}{2} g^{00} \delta^i{}_k (1 - e^{-2\frac{P_0}{\kappa}}) - i\delta^i{}_k g^{0s} P_s e^{-\frac{P_0}{\kappa}} +, \\
&\quad + i g^{0i} P_k (e^{-\frac{P_0}{\kappa}} - 1) + \frac{i}{2\kappa} \delta^i{}_k g^{rs} P_r P_s - \frac{i}{\kappa} g^{is} P_s P_k, \\
[P_\mu, P_\nu] &= 0, \\
[M^{\mu\nu}, M^{\lambda\sigma}] &= i(g^{\mu\sigma} M^{\nu\lambda} - g^{\nu\sigma} M^{\mu\lambda} + g^{\nu\lambda} M^{\mu\sigma} - g^{\mu\lambda} M^{\nu\sigma}).
\end{aligned}$$

The coproducts, counit and antipode:

$$\begin{aligned}
\Delta P_0 &= I \otimes P_0 + P_0 \otimes I, \\
\Delta P_k &= P_k \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_k, \\
\Delta M^{ij} &= M^{ij} \otimes I + I \otimes M^{ij}, \\
\Delta M^{i0} &= I \otimes M^{i0} + M^{i0} \otimes e^{-\frac{P_0}{\kappa}} - \frac{1}{\kappa} M^{ij} \otimes P_j, \\
\varepsilon(M^{\mu\nu}) &= 0; \quad \varepsilon(P_\nu) = 0; \quad \varepsilon(D) = 0, \\
S(P_0) &= -P_0, \\
S(P_i) &= -e^{\frac{P_0}{\kappa}} P_i, \\
S(M^{ij}) &= -M^{ij}, \\
S(M^{i0}) &= -e^{\frac{P_0}{\kappa}} (M^{i0} + \frac{1}{\kappa} M^{ij} P_j),
\end{aligned}$$

where $i, j, k = 1, 2, 3$.

Using, on the one hand, the properties of the left-invariant fields described by Woronowicz [5] and on the other hand, the duality relations $\mathcal{P}_\kappa \iff \tilde{\mathcal{P}}_\kappa$ established in [17], one can prove that the following substitutions:

$$\begin{aligned}
\chi_0 &= -i(\kappa(e^{\frac{P_0}{\kappa}} - 1) - \frac{g_{00}}{2\kappa} M^2), \\
\chi_i &= -i(e^{\frac{P_0}{\kappa}} P_i - g_{i0} \frac{M^2}{2\kappa}), \\
\chi &= -\frac{M^2}{8}, \\
\chi_{ij} &= -i(1 + \frac{i}{\kappa}\chi_0 - \frac{4}{\kappa^2} g_{00}\chi) M_{ij} - \frac{1}{\kappa}(\chi_0 M_{ij} + \chi_i M_{j0} - \chi_j M_{i0}), \\
\chi_{i0} &= -i(1 + \frac{i}{\kappa}\chi_0 - \frac{4}{\kappa^2} g_{00}\chi) M_{i0},
\end{aligned}$$

where

$$M^2 = g^{00}(2\kappa \sinh(\frac{P_0}{2\kappa}))^2 + 4\kappa g^{0l} P_l e^{\frac{P_0}{2\kappa}} \sinh(\frac{P_0}{2\kappa}) + g^{rs} P_r e^{\frac{P_0}{2\kappa}} P_s e^{\frac{P_0}{2\kappa}},$$

reproduce the algebra and coalgebra structure of our quantum Lie algebra.

Now, it is easy to see that the vectorfields χ and λ_μ are proportional to first casimir operator and deformed Paul-Lubaǹski invariant.

3 The $\kappa-$Poincarè group and algebra in case $g_{00} = 0$

After constructing this calculus, prof.J.Lukierski suggested me that the assumption that $g_{00} = 0$ should simplify this calculus. This problem in case of the differential calculus on $\kappa-$Minkowski space, is discussed in [6].It appears that in the case of

differential calculus on the $\kappa-$Poincarè group, under the assumption $g_{00} = 0$ we can obtain the differential calculus whose dimension is equal to the dimension of the classical differential calculus. In this case the ideal generated by the elements: $\Delta^{\alpha}{}_{\beta}\Delta^{\mu}{}_{\nu},\ \Delta^{\mu}{}_{\nu}{}^{\alpha},\ x^{\mu\nu}$ is adjoint invariant and is not equal to $ker\ \varepsilon$. This ideal gives us the ten-dimensional calculus and the dimension of square exterior power of our differential calculus is $\binom{10}{2}$ dimensional. But if $g_{00} \neq 0$ the situation dramatically changes, and we obtain the sexteen-dimensional differential calculus and the exterior power of our calculus need additional five differential forms.

To obtain the ideal $\mathcal{R}$ we put:

$$T \quad = \quad lin\{\ \Delta^{\mu}{}_{\nu}{}^{\alpha},\ \Delta^{\alpha}{}_{\beta}\Delta^{\mu}{}_{\nu},\ x^{\alpha\beta}\ \}.$$

In the definition of T we not subtract the trace of $x^{\mu\nu}$ and completely antisymmetric part of $\Delta^{\mu}{}_{\nu}{}^{\alpha}$.

One can prove that the linear set T is ad-invariant. Let $\mathcal{R}$ be a right ideal of $\mathcal{P}_{\kappa}$ generated by elements of T. Then the following theorem holds:

Theorem 2. $\mathcal{R}$ has the following properties:

(i) $\mathcal{R}$ is ad-invariant,

(ii) for any $a \in \mathcal{R},\ S(a)^* \in \mathcal{R}$,

(iii) $ker\ \varepsilon/\mathcal{R}$ is spanned by the following elements:

$$x^{\mu}; \quad \Delta^{\mu}{}_{\nu},\ \mu < \nu;$$

The basis of the space of the left-invariant 1-forms read:

$$\begin{aligned} \omega^{\mu} &= \pi r^{-1}(I \otimes x^{\mu}) = \Lambda_{\nu}{}^{\mu} dx^{\nu}, \\ \omega^{\mu}{}_{\nu} &= \pi r^{-1}(I \otimes \Delta^{\mu}{}_{\nu}) = \Lambda_{\alpha}{}^{\mu} d\Lambda^{\alpha}{}_{\nu}. \end{aligned} \tag{3.1}$$

The commutations rules between the invariant forms and generators of $\mathcal{P}_{\kappa}$ read:

$$\begin{aligned} [\Lambda^{\mu}{}_{\nu}, \omega^{\alpha}] &= -\frac{i}{\kappa}(g_{\nu 0}\Lambda^{\mu}{}_{\tau}\omega^{\tau\alpha} + \Lambda^{\mu}{}_{0}\omega^{\alpha}{}_{\nu}), \\ [x^{\mu}, \omega^{\alpha}] &= \frac{i}{\kappa}(\Lambda^{\mu\alpha}\omega_{0} - \delta^{\alpha}{}_{0}\Lambda^{\mu}{}_{\nu}\omega^{\nu}), \\ [\Lambda^{\alpha}{}_{\beta}, \omega^{\mu}{}_{\nu}] &= 0, \\ [x^{\alpha}, \omega^{\mu}{}_{\nu}] &= -\frac{i}{\kappa}(\delta^{\mu}{}_{0}\Lambda^{\alpha}{}_{\beta}\omega^{\beta}{}_{\nu} + g_{\nu 0}\Lambda^{\alpha}{}_{\beta}\omega^{\mu\beta} \\ &\quad -\Lambda^{\alpha}{}_{\nu}\omega^{\mu}{}_{0} - \Lambda^{\alpha\mu}\omega_{0\nu}). \end{aligned} \tag{3.2}$$

The external algebra read:

$$\begin{aligned} &\omega^{\mu}{}_{\nu} \wedge \omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} \wedge \omega^{\mu}{}_{\nu} = 0, \\ &\omega^{\alpha} \wedge \omega^{\mu} + \omega^{\mu} \wedge \omega^{\alpha} + \frac{i}{\kappa}(\delta^{\alpha}{}_{0}\omega^{\mu}{}_{\beta} \wedge \omega^{\beta} + \delta^{\mu}{}_{0}\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}) = 0, \\ &\omega^{\alpha} \wedge \omega^{\mu\nu} + \omega^{\mu\nu} \wedge \omega^{\alpha} + \frac{i}{\kappa}(\delta^{\mu}{}_{0}\omega^{\tau\nu} \wedge \omega_{\tau}{}^{\alpha} + \delta^{\nu}{}_{0}\omega^{\mu}{}_{\tau} \wedge \omega^{\tau\alpha} \\ &+\omega_{0}{}^{\nu} \wedge \omega^{\alpha\mu} + \omega^{\mu}{}_{0} \wedge \omega^{\alpha\nu}) = 0. \end{aligned} \tag{3.3}$$

The basis of $\Gamma^{\wedge 2}$ consists of the following elements:

$$\begin{array}{ll} \omega^{\alpha\beta}\wedge\omega^{\mu\nu}; & \text{for } \alpha<\beta,\ \mu<\nu,\ (\alpha\beta)\neq(\mu\nu),\ \alpha<\mu; \\ \omega^{\mu}\wedge\omega^{\nu},\ \omega^{\mu\nu}\wedge\omega^{\alpha}; & \text{for } \mu<\nu. \end{array}$$

The Cartan-Maurer equations have the following form:

$$\begin{aligned} d\omega^{\mu}{}_{\nu} &= \omega_{\tau}{}^{\mu}\wedge\omega^{\tau}{}_{\nu}, \\ d\omega^{\mu} &= \omega_{\tau}{}^{\mu}\wedge\omega^{\tau}. \end{aligned} \tag{3.4}$$

Note that this calculus can be obtained from the previous one by putting $g_{00} = 0,\ \Omega = \Omega_{\mu} = \mathcal{X}_{\mu} = \mathcal{Y} = 0$.

Introduceing the left-invariant fields, defined by the formula:

$$da = \frac{1}{2}(\chi_{\mu\nu} * a)\omega^{\mu\nu} + (\chi_{\mu} * a)\omega^{\mu},$$

we find the quantum Lie algebra:

$$\begin{aligned} [\chi_{\mu\nu},\chi_{\alpha}] &= (1+\frac{i}{\kappa}\chi_0)(\chi_{\mu}g_{\alpha\nu}-\chi_{\nu}g_{\alpha\mu}) \\ [\chi_{\alpha},\chi_{\mu}] &= 0 \\ [\chi_{\alpha\beta},\chi_{\mu\nu}] &= (\chi_{\beta\mu}g_{\nu\alpha}+\chi_{\alpha\nu}g_{\mu\beta}-\chi_{\beta\nu}g_{\mu\alpha}-\chi_{\alpha\mu}g_{\nu\beta}) \\ +\frac{i}{\kappa}(\chi_{\alpha}(\chi_{\mu0}g_{\nu\beta} &- \chi_{\nu0}g_{\mu\beta})+\chi_{\mu}(\chi_{\beta0}g_{\alpha\nu}-\chi_{\alpha0}g_{\nu\beta}) \\ +\chi_{\nu}(\chi_{\alpha0}g_{\mu\beta} &- \chi_{\beta0}g_{\alpha\mu})+\chi_{\beta}(\chi_{\nu0}g_{\alpha\mu}-\chi_{\mu0}g_{\alpha\nu})) \\ +\frac{i}{\kappa}(\chi_{\beta}(\chi_{\mu\alpha}g_{0\nu} &- \chi_{\nu\alpha}g_{0\mu})+\chi_{\nu}(\chi_{\beta\mu}g_{0\alpha}-\chi_{\alpha\mu}g_{0\beta}) \\ +\chi_{\mu}(\chi_{\alpha\nu}g_{0\beta} &- \chi_{\beta\nu}g_{0\alpha})+\chi_{\alpha}(\chi_{\nu\beta}g_{0\mu}-\chi_{\mu\beta}g_{0\nu}). \end{aligned}$$

One can prove that the following substitutions:

$$\begin{aligned} \chi_0 &= -i\kappa(e^{\frac{P_0}{\kappa}}-1), \\ \chi_i &= -i(e^{\frac{P_0}{\kappa}}P_i - g_{i0}\frac{M^2}{2\kappa}), \\ \chi_{ij} &= -i(1+\frac{i}{\kappa}\chi_0)M_{ij}-\frac{1}{\kappa}(\chi_0 M_{ij}+\chi_i M_{j0}-\chi_j M_{i0}), \\ \chi_{i0} &= -i(1+\frac{i}{\kappa}\chi_0)M_{i0}, \end{aligned}$$

where

$$\begin{aligned} M^2 &= g^{00}(2\kappa\sinh(\frac{P_0}{2\kappa}))^2+4\kappa g^{0l}P_l e^{\frac{P_0}{2\kappa}}\sinh(\frac{P_0}{2\kappa}) \\ &\quad +g^{rs}P_r e^{\frac{P_0}{2\kappa}}P_s e^{\frac{P_0}{2\kappa}}, \end{aligned}$$

reproduce the algebra and coalgebra structure of our quantum Lie algebra.

4 The $\kappa-$Weyl group and algebra

The Weyl group $\mathcal{W}$ consists of the triples (x, Λ, e^b), where x is a $4-$vector, Λ is the matrix of the Lorentz group in $4-$dimensions and $b \in R$, with the composition law:

$$(x^\mu, \Lambda^\mu{}_\nu, e^b) * (x'^\nu, \Lambda'^\nu{}_\alpha, e^{b'}) = (\Lambda^\mu{}_\nu e^b x'^\nu + x^\mu, \Lambda^\mu{}_\nu \Lambda'^\nu{}_\alpha, e^b e^{b'}).$$

In [1], under assumption that $g_{00} = 0$ the $\kappa-$deformation of the Weyl group $\mathcal{W}_\kappa$ was constructed. This $\kappa-$Weyl group $\mathcal{W}_\kappa$ is a Hopf $*-$algebra defined as follows [1]. Consider the universal $*-$algebra with unity generated by self adjoint elements $\Lambda^\mu{}_\nu,\ x^\mu$ and e^b subject to the following relations:

$$\begin{aligned}
[\Lambda^\alpha{}_\beta, x^\varrho] &= -\frac{i}{\kappa}((e^b \Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\varrho{}_\beta + (\Lambda_{0\beta} - e^b g_{0\beta}) g^{\alpha\varrho}), \\
[x^\varrho, x^\sigma] &= \frac{i}{\kappa}(\delta^\varrho{}_0 x^\sigma - \delta^\sigma{}_0 x^\varrho), \\
[\Lambda^\alpha{}_\beta, \Lambda^\mu{}_\nu] &= 0, \\
[\Lambda^\mu{}_\nu, e^b] &= 0, \\
[x^\mu, e^b] &= 0.
\end{aligned}$$

The coproduct, antipode and counit are defined as follows:

$$\begin{aligned}
\Delta \Lambda^\mu{}_\nu &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, \\
\Delta x^\mu &= e^b \Lambda^\mu{}_\nu \otimes x^\nu + x^\mu \otimes I, \\
\Delta e^b &= e^b \otimes e^b, \\
S(\Lambda^\mu{}_\nu) &= \Lambda_\nu{}^\mu, \\
S(x^\mu) &= -e^{-b} \Lambda_\nu{}^\mu x^\nu, \\
S(e^b) &= e^{-b}, \\
\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu, \\
\varepsilon(x^\mu) &= 0, \\
\varepsilon(e^b) &= 1.
\end{aligned}$$

In order to obtain the bicovariant $*-$calculi we go along the same lines as in the $\kappa-$Poincare group.

Let $\mathcal{R}$ be a right ideal of $\mathcal{W}_\kappa$ generated by the following elements: $(e^b - I)^2,\ \Delta^\mu{}_\nu{}^\alpha,\ \Delta^\mu{}_\nu(e^b - I),\ x^\alpha(e^b - I),\ \Delta^\alpha{}_\beta \Delta^\mu{}_\nu,\ x^{\alpha\beta}$. Then we have the following:

Theorem 3. $\mathcal{R}$ has the following properties:

(i) $\mathcal{R}$ is ad-invariant,

(ii) for any $a \in \mathcal{R},\ S(a)^* \in \mathcal{R}$,

(iii) $ker\ \varepsilon/\mathcal{R}$ is spanned by the following elements:

$$x^\mu; \quad \Delta^\mu{}_\nu,\ \mu < \nu; \quad e^b - I.$$

We see that our calculus is eleven-dimensional. The basis of the space of the left-invariant 1-forms reads:

$$\begin{aligned}\omega^\mu &= \pi r^{-1}(I\otimes x^\mu) = e^{-b}\Lambda_\nu{}^\mu dx^\nu,\\ \omega^\mu{}_\nu &= \pi r^{-1}(I\otimes\Delta^\mu{}_\nu) = \Lambda_\alpha{}^\mu d\Lambda^\alpha{}_\nu,\\ \omega^b &= \pi r^{-1}(I\otimes(e^b-I)) = e^{-b}de^b.\end{aligned}\tag{4.1}$$

The commutations rules between the invariant forms and generators of $\mathcal{W}_\kappa$ read:

$$\begin{aligned}[][e^b,\omega^\mu] &= 0, \qquad [e^b,\omega^\mu{}_\nu]=0, \qquad [e^b,\omega^b]=0,\\ [\Lambda^\mu{}_\nu,\omega^b] &= 0, \qquad [x^\alpha,\omega^b]=0,\\ [\Lambda^\mu{}_\nu,\omega^\alpha] &= \frac{i}{\kappa}(g_{\nu 0}\Lambda^{\mu\alpha}-\delta^\alpha{}_\nu\Lambda^\mu{}_0)\omega^b\\ &\quad -\frac{i}{\kappa}(g_{\nu 0}\Lambda^\mu{}_\tau\omega^{\tau\alpha}+\Lambda^\mu{}_0\omega^\alpha{}_\nu),\\ [x^\mu,\omega^\alpha] &= \frac{i}{\kappa}e^b(\Lambda^{\mu\alpha}\omega_0-\delta^\alpha_0\Lambda^\mu{}_\nu\omega^\nu),\\ [\Lambda^\alpha{}_\beta,\omega^\mu{}_\nu] &= 0,\\ [x^\alpha,\omega^\mu{}_\nu] &= -\frac{i}{\kappa}e^b(\delta^\mu{}_0\Lambda^\alpha{}_\beta\omega^\beta{}_\nu+g_{\nu 0}\Lambda^\alpha{}_\beta\omega^{\mu\beta}\\ &\quad -\Lambda^\alpha{}_\nu\omega^\mu{}_0-\Lambda^{\alpha\mu}\omega_{0\nu}).\end{aligned}\tag{4.2}$$

The external power $\Gamma^{\wedge 2}$ is described by the following relations:

$$\begin{aligned}&\omega^b\wedge\omega^b=0,\\ &\omega^\mu{}_\nu\wedge\omega^\alpha{}_\beta+\omega^\alpha{}_\beta\wedge\omega^\mu{}_\nu=0,\\ &\omega^b\wedge\omega^\mu{}_\nu+\omega^\mu{}_\nu\wedge\omega^b=0,\\ &\omega^b\wedge\omega^\mu+\omega^\mu\wedge\omega^b=0,\\ &\omega^\alpha\wedge\omega^\mu+\omega^\mu\wedge\omega^\alpha+\frac{i}{\kappa}(\delta^\alpha{}_0\omega^\mu{}_\beta\wedge\omega^\beta+\delta^\mu{}_0\omega^\alpha{}_\beta\wedge\omega^\beta\\ &+\delta^\mu{}_0\omega^b\wedge\omega^\alpha+\delta^\alpha{}_0\omega^b\wedge\omega^\mu)+\frac{2i}{\kappa}g^{\mu\alpha}\omega_0\wedge\omega^b=0,\\ &\omega^\alpha\wedge\omega^{\mu\nu}+\omega^{\mu\nu}\wedge\omega^\alpha+\frac{i}{\kappa}(\delta^\mu{}_0\omega^{\tau\nu}\wedge\omega_\tau{}^\alpha+\delta^\nu{}_0\omega^\mu{}_\tau\wedge\omega^{\tau\alpha}\\ &+\omega_0{}^\nu\wedge\omega^{\alpha\mu}+\omega^\mu{}_0\wedge\omega^{\alpha\nu})+\frac{i}{\kappa}(\delta^\mu{}_0\omega^b\wedge\omega^{\alpha\nu}+\delta^\nu{}_0\omega^b\wedge\omega^{\mu\alpha}\\ &+g^{\alpha\mu}\omega_0{}^\nu\wedge\omega^b+g^{\alpha\nu}\omega^\mu{}_0\wedge\omega^b)=0.\end{aligned}\tag{4.3}$$

The basis of $\Gamma^{\wedge 2}$ consists of the following elements:

$$\begin{aligned}&\omega^{\alpha\beta}\wedge\omega^{\mu\nu}; \quad \text{for } \alpha<\beta,\ \mu<\nu,\ (\alpha\beta)\neq(\mu\nu),\ \alpha<\mu;\\ &\omega^{\mu\nu}\wedge\omega^b,\ \omega^\mu\wedge\omega^\nu,\ \omega^{\mu\nu}\wedge\omega^\alpha; \quad \text{for } \mu<\nu;\\ &\omega^\mu\wedge\omega^b.\end{aligned}$$

The Cartan-Maurer equations have the following form:

$$\begin{aligned} d\omega^{\mu}{}_{\nu} &= \omega_{\tau}{}^{\mu} \wedge \omega^{\tau}{}_{\nu}, \\ d\omega^{\mu} &= \omega^{\mu} \wedge \omega^{b} + \omega_{\tau}{}^{\mu} \wedge \omega^{\tau}, \\ d\omega^{b} &= 0. \end{aligned} \tag{4.4}$$

Note that if we put $b = 0$ in eq.(4.1), (4.2), we obtain the same sets of relations like in eq.(3.1), (3.2), but it is not true for the case of eq.(4.3), (4.4) end eq.(3.3), (3.4).

In order to obtain the quantum Lie algebra we introduce the left-invariant field defined by the formula:

$$da = \frac{1}{2}(\chi_{\mu\nu} * a)\omega^{\mu\nu} + (\chi_{\mu} * a)\omega^{\mu} + (\lambda * a)\omega^{b}.$$

The resulting quantum Lie algebra reads:

$$\begin{aligned} [\chi_{\mu\nu}, \chi_{\alpha}] &= (1 + \frac{i}{\kappa}\chi_0)(\chi_{\mu} g_{\alpha\nu} - \chi_{\nu} g_{\alpha\mu}), \\ [\chi_{\alpha}, \chi_{\mu}] &= 0 \\ [\chi_{\mu}, \lambda] &= -\chi_{\mu} - \frac{i}{\kappa}\chi_0\chi_{\mu} + \frac{i}{\kappa} g_{0\mu} g^{\sigma\alpha}\chi_{\sigma}\chi_{\alpha}, \\ [\chi_{\mu\nu}, \lambda] &= -\frac{i}{\kappa}(\chi_{\mu}\chi_{0\nu} + \chi_{\nu}\chi_{\mu 0}) \\ &\quad + \frac{i}{\kappa} g^{\alpha\sigma}(\chi_{\alpha}\chi_{\sigma\nu} g_{0\mu} + \chi_{\alpha}\chi_{\mu\sigma} g_{0\nu}), \\ [\chi_{\alpha\beta}, \chi_{\mu\nu}] &= (\chi_{\beta\mu} g_{\nu\alpha} + \chi_{\alpha\nu} g_{\mu\beta} - \chi_{\beta\nu} g_{\mu\alpha} - \chi_{\alpha\mu} g_{\nu\beta}) \\ + \frac{i}{\kappa}(\chi_{\alpha}(\chi_{\mu 0} g_{\nu\beta} &- \chi_{\nu 0} g_{\mu\beta}) + \chi_{\mu}(\chi_{\beta 0} g_{\alpha\nu} - \chi_{\alpha 0} g_{\nu\beta}) \\ + \chi_{\nu}(\chi_{\alpha 0} g_{\mu\beta} &- \chi_{\beta 0} g_{\alpha\mu}) + \chi_{\beta}(\chi_{\nu 0} g_{\alpha\mu} - \chi_{\mu 0} g_{\alpha\nu})) \\ + \frac{i}{\kappa}(\chi_{\beta}(\chi_{\mu\alpha} g_{0\nu} &- \chi_{\nu\alpha} g_{0\mu}) + \chi_{\nu}(\chi_{\beta\mu} g_{0\alpha} - \chi_{\alpha\mu} g_{0\beta}) \\ + \chi_{\mu}(\chi_{\alpha\nu} g_{0\beta} &- \chi_{\beta\nu} g_{0\alpha}) + \chi_{\alpha}(\chi_{\nu\beta} g_{0\mu} - \chi_{\mu\beta} g_{0\nu})). \end{aligned}$$

The Weyl algebra reads [1]:
The commutation rules:

$$\begin{aligned} [M^{ij}, P_0] &= 0 \\ [M^{ij}, P_k] &= i\kappa(\delta^{j}{}_{k} g^{0i} - \delta^{i}{}_{k} g^{0j})(1 - e^{-\frac{P_0}{\kappa}}) + i(\delta^{j}{}_{k} g^{is} - \delta^{i}{}_{k} g^{js})P_s \\ [M^{i0}, P_0] &= i\kappa g^{i0}(1 - e^{-\frac{p_0}{\kappa}}) + i g^{ik} P_k \\ [M^{i0}, P_k] &= -i\frac{\kappa}{2} g^{00}\delta^{i}{}_{k}(1 - e^{-2\frac{P_0}{\kappa}}) - i\delta^{i}{}_{k} g^{0s} P_s e^{-\frac{P_0}{\kappa}} + \\ &\quad + i g^{0i} P_k(e^{-\frac{P_0}{\kappa}} - 1) + \frac{i}{2\kappa}\delta^{i}{}_{k} g^{rs} P_r P_s - \frac{i}{\kappa} g^{is} P_s P_k \\ [P_{\mu}, P_{\nu}] &= 0, \\ [M^{\mu\nu}, M^{\lambda\sigma}] &= i(g^{\mu\sigma} M^{\nu\lambda} - g^{\nu\sigma} M^{\mu\lambda} + g^{\nu\lambda} M^{\mu\sigma} - g^{\mu\lambda} M^{\nu\sigma}), \end{aligned}$$

$$
\begin{aligned}
[M_{\mu\nu}, D] &= 0, \\
[D, P_0] &= i\kappa(1 - e^{-\frac{P_0}{\kappa}}) \\
[D, P_i] &= iP_i e^{-\frac{P_0}{\kappa}} + i\frac{\kappa}{2} g^{00} g_{i0}(1 - e^{-\frac{P_0}{\kappa}})^2 + \\
&\quad + i g_{0i} g^{0s} P_s (1 - e^{-\frac{P_0}{\kappa}}) + \frac{i}{2\kappa} g_{0i} g^{rs} P_r P_s
\end{aligned}
$$

The coproducts, counit and antipode:

$$
\begin{aligned}
\Delta D &= D \otimes I + I \otimes D - g_{0i} M^{i0} \otimes (1 - e^{-\frac{P_0}{\kappa}}) - \frac{1}{\kappa} g_{0i} M^{ik} \otimes P_k \\
\Delta P_0 &= I \otimes P_0 + P_0 \otimes I \\
\Delta P_k &= P_k \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_k \\
\Delta M^{ij} &= M^{ij} \otimes I + I \otimes M^{ij} \\
\Delta M^{i0} &= I \otimes M^{i0} + M^{i0} \otimes e^{-\frac{P_0}{\kappa}} - \frac{1}{\kappa} M^{ij} \otimes P_j, \\
\varepsilon(M^{\mu\nu}) &= 0; \quad \varepsilon(P_\nu) = 0; \quad \varepsilon(D) = 0, \\
S(P_0) &= -P_0, \\
S(P_i) &= -e^{\frac{P_0}{\kappa}} P_i, \\
S(M^{ij}) &= -M^{ij}, \\
S(M^{i0}) &= -e^{\frac{P_0}{\kappa}} (M^{i0} + \frac{1}{\kappa} M^{ij} P_j), \\
S(D) &= -D + g_{i0} M^{i0} - g_{i0} e^{\frac{P_0}{\kappa}} M^{i0} - \frac{1}{\kappa} g_{i0} e^{\frac{P_0}{\kappa}} M^{ik} P_k.
\end{aligned}
$$

Using, on the one hand, the properties of the left-invariant fields described by Woronowicz [5] and on the other hand, the duality relations $\mathcal{W}_\kappa \iff \tilde{\mathcal{W}}_\kappa$ established in [17], one can prove that the following substitutions:

$$
\begin{aligned}
\chi_0 &= -i\kappa(e^{\frac{P_0}{\kappa}} - 1), \\
\chi_i &= -i(e^{\frac{P_0}{\kappa}} P_i - g_{i0} \frac{M^2}{2\kappa}), \\
\chi_{ij} &= -i(1 + \frac{i}{\kappa}\chi_0) M_{ij} - \frac{1}{\kappa}(\chi_0 M_{ij} + \chi_i M_{j0} - \chi_j M_{i0}), \\
\chi_{i0} &= -i(1 + \frac{i}{\kappa}\chi_0) M_{i0}, \\
\lambda &= -iD - \frac{1}{\kappa} \chi^\alpha M_{\alpha 0},
\end{aligned}
$$

where

$$
\begin{aligned}
M^2 &= g^{00}(2\kappa \sinh(\frac{P_0}{2\kappa}))^2 + 4\kappa g^{0l} P_l e^{\frac{P_0}{2\kappa}} \sinh(\frac{P_0}{2\kappa}) \\
&\quad + g^{rs} P_r e^{\frac{P_0}{2\kappa}} P_s e^{\frac{P_0}{2\kappa}},
\end{aligned}
$$

reproduce the algebra and coalgebra structure of our quantum Lie algebra.

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References

[1] P.Kosiński, P.Maślanka *The $\kappa-$Weyl group and its algebra* in "From Field Theory to Quantum Groups" vol. on 60 anniversary of J.Lukierski, World Scientific Singapur, 1996 or **Q-ALG** 9512018.

[2] P.Kosiński, P.Maślanka, Jan Sobczyk *The bicovariant diffrerential calculus on the $\kappa-$Poincarè group and on the $\kappa-$Minkowski space* Czechoslovak J. Phys. **46** (1996), 201-208 or **Q-ALG** 9508021.

[3] P.Kosiǹski, M.Majewski, P.Maŝlanka *The bicovariant differential calculus on the three-dimensional Kappa-Poincar'e group* **Q-ALG** 9602004, in print in Acta Phys. Pol. **B** .

[4] C.Gonera, P.Kosiǹski, P.Maŝlanka *Differential calculi on quantum Minkowski space* **Q-ALG** 9602007, in print in J. Math. Phys.

[5] S.L.Woronowicz *Differential Calculus on Compact Matrix Pseudogroups (Quantums Groups)* Comm. Math. Phys. **122** (1989), 125.

[6] P.Kosiński, J.Lukierski, P.Maślanka, A.Sitarz -paper in preparation.

[7] W.B.Schmidke, J.Weiss, B.Zumino Zeitschr. f. Physik **52** (1991), 472.

U.Carow-Watamura, M.Schliecker, M.Scholl, S.Watamura Int. J. Mod. Phys. **A 6** (1991), 3081.

S.L. Woronowicz Comm. Math. Phys. **136** (1991), 399.

O. Ogievetsky, W.B. Schmidke, J. Weiss, B. Zumino Comm. Math. Phys. **150** (1992), 495.

M. Chaichian, A.P. Demichev Proceedings of the Workshop: "Generalized symmetries in Physics", Clausthal (1993), .

V. Dobrev J. Phys. **A 26** (1993), 1317.

L. Castellani in "Quantum Groups" Proceedings of XXX Karpacz Winter School of Theoretical Physics, Karpacz 1994, PWN 1995, p. 13 .

[8] J. Lukierski, A. Nowicki, H. Ruegg, V. Tolstoy Phys. Lett. **B 264** (1991), 331.

J. Lukierski, A. Nowicki, H. Ruegg Phys. Lett. **B 293** (1993), 344.

S. Giller, P. Kosiński, J. Kunz, M. Majewski, P. Maślanka Phys. Lett. **B 286** (1992), 57.

[9] S. Zakrzewski J. Phys. **A 27** (1994), 2075.

[10] J. Lukierski, H. Ruegg Phys. Lett. **B 329** (1994), 189.

P. Maślanka J. Phys. **A 26** (1993), L1251.

[11] S. Giller, P. Kosiński, J. Kunz, M. Majewski, P. Maślanka Phys. Lett. **B 286** (1992), 57.

S. Giller, C. Gonera, P. Kosiński, J. Kunz, P. Maślanka Mod. Phys. Lett. **A 8** (1993), 3785.

J. Lukierski, H. Ruegg, W. Rühl Phys. Lett. **B 313** (1993), 357.

[12] P. Maślanka J. Math. Phys. **34** (1993), 6025.

[13] ——— J. Phys. **A 27** (1994), 7099.

[14] S. Majid Pacific J. Math. **141** (1990), 311.

——— Int. J. Mod. Phys. **A 5** (1990), 1.

——— J. Algebra **130** (1990), 17.

[15] ———, H. Ruegg Phys. Lett. **B 334** (1994), 348.

Ph. Zaugg preprint MIT–CTP–2353 (1994), .

[16] A. Ballesteros, E. Celeghini, R. Giachetti, E. Sorace, M. Tarlini J. Phys. **A 26** (1993), 7495.

P. Maślanka J. Math. Phys. **35** (1994), 1976.

Ph. Zaugg preprint MIT–CTP–2353 (1994), .

[17] P. Kosiński, P. Maślanka *preprint IMUŁ 3/94* or **Q-ALG** 9411033.